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Inefficiency of Logit-Based Stochastic User Equilibrium in a Traffic Network Under ATIS

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Efficiency Losses of Logit-based Stochastic User Equilibria in Traffic Networks under ATIS

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Abstract In this paper we derive the upper bounds of the cost inefficiencies of logit-based stochastic user equilibrium (SUE) traffic flow patterns under Advanced Traveler Information Systems (ATIS). All drivers are divided into two groups, one equipped with ATIS and another without, and both of which follow the logit-based SUE principle in route choice. The equipped drivers have less degree of travel time variability than the unequipped ones. The cost inefficiency is defined as a ratio between the total cost incurred by players in a SUE state and the minimum-possible total cost. The effects of various parameters on the bound are investigated. It is found that the increasing of congestion degree, total demand and network complexity will make the bound go up, while the promotion of ATIS market penetration and information quality will reduce the bound.

Keywords Stochastic user equilibrium; System optimum; Efficiency loss; ATIS market penetration; Network

1 Introduction

Recently, there has been a surge of interest in transportation science on the consequences of selfish travel behavior. For quantifying the degradation in network performance due to the lack of coordination, Koutsoupias and Papadimitriou (1999) introduced the notion of coordination ratio or price of anarchy to measure the worst-case performance of uncoordinated users against the optimal solution achieved by centralized control. The term

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"price of anarchy" has been coined to characterize the degree of inefficiency which is the worst possible ratio between the total cost incurred by players in a Nash equilibrium situation and the minimum-possible total cost. The minimum-possible cost is generally implemented by centralized control. Roughgarden (2002) proved that the worst-case inefficiency due to selfish routing is independent of the network topology. Roughgarden and Tardos (2002, 2004) further proved that the total time of a non-toll user equilibrium (UE) is at most that of the system optimum (SO) with doubled demand in the same network. Moreover, the total time of a non-toll UE is at most 4/3 times that of the SO when all link travel time functions are linear. Correa et al. (2004) provided a simple geometric proof for the price of anarchy for the user equilibrium with fixed demand, using a variational inequality (VI) expression of UE conditions. They also showed that the addition of link capacity constraints does not change the worst ratio between the best UE and the SO. Chau and Sim (2003) proved that the price of anarchy result still holds for systems with symmetric non-separable link travel time functions. The efficiency of atomic splittable selfish route with polynomial cost functions was studied by Yang et al. (2008). Roughgarden (2005) summarized the latest developments of this research subject.

In the context of traffic networks, the studies up to date in deriving the inefficiency upper bounds contain such an assumption that the selfish drivers have perfect information about the network's traffic condition and know other drivers' route choices, so the deterministic UE state being reached. The interpretation of the stochastic user equilibrium (SUE) in the literature is that it is an outcome of all users' route choices for minimizing their perceived travel costs (Sheffi, 1984). The perceived travel cost is regarded as a random variable due to travel time uncertainty and users' perception error of travel time. Following this line of thinking, Guo and Yang (2005) and Yang and Huang (2005) investigated the bound of the cost inefficiency of a SUE flow pattern in relation to the Wardropian system optimum.

The advanced traveler information systems (ATIS) that provide travelers real-time information of traffic condition, are generally believed to be efficient in many aspects such as improving individuals' trip planning, alleviating road congestion and enhancing network performance. For investigating the multiple equilibrium behaviors of different classes of users, various multi-class traffic models have been developed, which differentiate travelers who receive information versus those who don't have information (Harker, 1988; Liu et al., 2007; Van Vuren and Watling, 1991; Yang, 1998).

In this paper, we extend the works of Guo and Yang (2005) and Yang and Huang (2005) that dealt with single user class only, to the two-user class case in which one class is equipped with ATIS and another without. All drivers are divided into two classes, both of which follow the SUE principle in route choices. Drivers who are equipped with ATIS are able to receive real-time traffic information so as to assess the routes' attributes more accurately than unequipped drivers. This means equipped drivers have less degree of travel time variability than unequipped drivers. We derive the upper bounds of the cost inefficiencies caused by stochastic user equilibrium with two user classes and examine the effects of various parameters on the bound. These parameters are the ones reflecting the network's complexity, the link cost function's property, the ATIS quality and market penetration.

2 The SO and two-user class SUE models

Consider a network $G=(N,A)$, where N is the set of nodes and A the set of links in the network. Let W be the set of all OD (origin-destination) pairs and R_w the set of all possible routes connecting OD pair $w \in W$. Considering traffic congestion, we assume that the travel time on each link $a \in A$, is a monotonically increasing and convex function of the link flow v_a. This function is also separable, we then have $t_a = t_a(v_a)$, $a \in A$.

2.1 The system optimum

The Wardropian SO model that minimizes the system's total travel time is expressed by

$$\min_{\mathbf{v}\in\Omega_v} T(\mathbf{v}) = \sum_{a\in A} t_a(v_a)v_a \tag{1}$$

where Ω_v is defined by

$$\Omega_v = \left\{ \mathbf{v} \middle| \mathbf{v} = \mathbf{\Delta f}, \mathbf{\Lambda f} = \mathbf{d}, \mathbf{f} \geq 0 \right\} \tag{2}$$

where $\mathbf{v}=(v_a, a\in A)^{\mathrm{T}}$, $\mathbf{d}=(d_w, w\in W)^{\mathrm{T}}$ and $\mathbf{f}=(f_{rw}, r\in R_w, w\in W)^{\mathrm{T}}$, representing the vectors of link flows, OD demands and path flows, respectively; $\mathbf{\Delta}=[\delta_{ar}]$ is the link/path incidence matrix, δ_{ar} = {1, if link a is on path r and 0, otherwise}; $\mathbf{\Lambda}=[\Lambda_{rw}]$ is the OD/path incidence matrix, Λ_{rw} = {1, if path $r\in R_w$ and 0, otherwise}. Since $t_a(v_a)$ is monotonically increasing and convex, then $t_a(v_a)v_a$ is convex and thus, the solution to the SO problem (1)-(2) is unique.

2.2 The two-user class SUE with ATIS market penetration

In reality, the information available to equipped drivers could be partial or imperfect, and thus the travel time perceived by them, although with help from ATIS, is stochastic, but with less variability than that by unequipped drivers. Let $\lambda_w d_w$ and $(1-\lambda_w)d_w$ be the demands of the equipped and unequipped drivers between OD pair $w \in W$, respectively, here λ_w ($0 \le \lambda_w \le 1$) is the ATIS market penetration between OD pair $w \in W$. Suppose that all random terms representing the travel time uncertainty and users' perception error, are independent and identical distributed Gumbel variables with mean zero, according to the utility maximization theory, at equilibrium the path choices of equipped and unequipped drivers are the solution of the following convex minimization problem (Huang and Li, 2007)

$$\min Z\left(\mathbf{f},\overline{\mathbf{f}}\right) = \sum_{a\in A}\int_0^{v_a} t_a(x)\,\mathrm{d}x + \frac{1}{\theta}\sum_{w\in W}\sum_{r\in R_w} f_{rw}\ln f_{rw} + \frac{1}{\overline{\theta}}\sum_{w\in W}\sum_{r\in R_w} \overline{f}_{rw}\ln \overline{f}_{rw} \tag{3}$$

subject to

$$\sum_{r\in R_w} f_{rw} = \lambda_w d_w,\quad w\in W \tag{4}$$

$$\sum_{r\in R_w} \overline{f}_{rw} = (1-\lambda_w)d_w,\quad w\in W \tag{5}$$

$$f_{rw} \ge 0,\quad r\in R_w,\quad w\in W \tag{6}$$

$$\overline{f}_{rw} \ge 0,\quad r\in R_w,\quad w\in W \tag{7}$$

$$v_a = \sum_{w\in W}\sum_{r\in R_w}\left(f_{rw}+\overline{f}_{rw}\right)\delta_{ar},\quad a\in A \tag{8}$$

where f_{rw} and $\overline{f}_{rw}$ are the path flows of equipped and unequipped drivers on path $r \in R_w$ between OD pair $w \in W$, respectively. θ and $\overline{\theta}$ are the parameters representing the travel time variations of equipped and unequipped drivers, respectively. A higher θ-value means a smaller variation for the equipped drivers. Hence, the θ-value can be used to represent the quality of the traffic information provided by ATIS. The parameter $\overline{\theta}$ reflects the familiarity degree to traffic conditions by unequipped drivers. The relation $\theta > \overline{\theta}$ holds, which claims that the equipped drivers have lower variations on travel time than the unequipped.

Solving the above minimization problem (3)-(8) gives the following logit-based path flow solution

$$f_{rw} = \lambda_w d_w \frac{\exp\left(-\theta c_{rw}\right)}{\sum_{l \in R_w} \exp\left(-\theta c_{lw}\right)}, \quad r \in R_w, w \in W \tag{9}$$

$$\bar{f}_{rw} = \left(1-\lambda_w\right) d_w \frac{\exp\left(-\bar{\theta} c_{rw}\right)}{\sum_{l \in R_w} \exp\left(-\bar{\theta} c_{lw}\right)}, \quad r \in R_w, w \in W \tag{10}$$

where c_{rw} is the actual (or measured) travel time of path $r \in R_w$ between OD pair $w \in W$, $c_{rw} = \sum_{a \in A} t_a\left(v_a\right)\delta_{ar}$. The existence and uniqueness of the path flow solution are guaranteed since all link travel time functions in the problem (3)-(8) are monotonically increasing and the feasible region is compact (Sheffi, 1984).

3 Determination of the cost inefficiency bound

Let $\mathbf{f}^{\text{so}}$ and $\mathbf{v}^{\text{so}}$ be the path flow and link flow solutions of the SO problem (1)-(2) respectively, $\left(\mathbf{f}^{\text{sue}}, \bar{\mathbf{f}}^{\text{sue}}\right)$ and $\mathbf{v}^{\text{sue}}$ be the path flow and link flow solutions of the SUE problem (3)-(8) respectively. We define the following ratio

$$\rho_{\text{mixed}}^{\text{sue}} = \frac{T_{\text{mixed}}^{\text{sue}}}{T^{\text{so}}} = \frac{T\left(\mathbf{v}^{\text{sue}}\right)}{T\left(\mathbf{v}^{\text{so}}\right)} \tag{11}$$

where $T_{\text{mixed}}^{\text{sue}} = T\left(\mathbf{v}^{\text{sue}}\right) = \sum_{a \in A} t_a\left(v_a^{\text{sue}}\right) v_a^{\text{sue}}$ with $v_a^{\text{sue}} = \sum_{w \in W} \sum_{r \in R_w} \left(f_{rw}^{\text{sue}} + \bar{f}_{rw}^{\text{sue}}\right)\delta_{ar}$, $a \in A$, and $T^{\text{so}} = T\left(\mathbf{v}^{\text{so}}\right) = \sum_{a \in A} t_a\left(v_a^{\text{so}}\right) v_a^{\text{so}}$ with $v_a^{\text{so}} = \sum_{w \in W} \sum_{r \in R_w} f_{rw}^{\text{so}} \delta_{ar}$, $a \in A$. Clearly, $\rho_{\text{mixed}}^{\text{sue}} \geq 1$ always holds since selfish routing generally does not yield an SO flow pattern (i.e., the UE and SUE are typically inefficient). In this study, we call the ratio $\rho_{\text{mixed}}^{\text{sue}}$ the cost inefficiency, or the price of anarchy, of the two-user class stochastic user equilibrium under ATIS. The purpose of our study is to find the upper bound of the inefficiency $\rho_{\text{mixed}}^{\text{sue}}$. For this, we first introduce a lemma.

Lemma 1 *If the separable link cost function,* $t_a\left(v_a\right)$, $a \in A$, *is monotonically increasing* *with link flow, a logit-based two-user class SUE problem with fixed OD demand is equivalent to the following variational inequality, i.e., find* $\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\in\Omega_{f,\bar{f}}$ *satisfying Eqs.(4)-(7) , such that*

$$\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)+\frac{1}{\theta}\ln f_{rw}^{\text{sue}}\right)\left(f_{rw}-f_{rw}^{\text{sue}}\right)$$

$$+\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)+\frac{1}{\overline{\theta}}\ln \overline{f}_{rw}^{\text{sue}}\right)\left(\overline{f}_{rw}-\overline{f}_{rw}^{\text{sue}}\right)\geq 0,\ \left(\mathbf{f},\overline{\mathbf{f}}\right)\in\Omega_{f,\bar{f}} \tag{12}$$

where $\theta>0$ *and* $\overline{\theta}>0$ *are the travel time variability parameters of the two user classes respectively,* $c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)=\sum_{a\in A}t_a\left(v_a^{\text{sue}}\right)\delta_{ar}$ *is the travel cost of path* r *between OD pair* w*, and*

$$\Omega_{f,\bar{f}}=\left\{\left(\mathbf{f},\overline{\mathbf{f}}\right)\middle|\boldsymbol{\Lambda}\mathbf{f}=\boldsymbol{\lambda}\mathbf{d},\ \boldsymbol{\Lambda}\overline{\mathbf{f}}=\left(1-\boldsymbol{\lambda}\right)\mathbf{d},\mathbf{f}\geq 0,\overline{\mathbf{f}}\geq 0\right\} \tag{13}$$

Proof The problem (3)-(8) has unique path flow solution $\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\in\Omega_{f,\bar{f}}$, and its entropy-type function ensures that the optimum is achieved at an interior point. Using $v_a=\sum_{w\in W}\sum_{r\in R_w}\left(f_{rw}+\overline{f}_{rw}\right)\delta_{ar}$, a necessarily and sufficient condition for a unique optimum $\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\in\Omega_{f,\bar{f}}$ is that

$$\left[\nabla_{\mathbf{f}}Z\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\right]^{\mathrm{T}}\left(\mathbf{f}-\mathbf{f}^{\text{sue}}\right)+\left[\nabla_{\overline{\mathbf{f}}}Z\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\right]^{\mathrm{T}}\left(\overline{\mathbf{f}}-\overline{\mathbf{f}}^{\text{sue}}\right)\geq 0,$$

$$\left(\mathbf{f},\overline{\mathbf{f}}\right)\in\Omega_{f,\bar{f}} \tag{14}$$

Substituting

$$\left[\nabla_{\mathbf{f}}Z\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\right]^{\mathrm{T}}=\left[\cdots,\frac{1}{\theta}\left(1+\ln f_{rw}^{\text{sue}}\right)+c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right),\cdots\right]^{\mathrm{T}}$$

and

$$\left[\nabla_{\overline{\mathbf{f}}}Z\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)\right]^{\mathrm{T}}=\left[\cdots,\frac{1}{\overline{\theta}}\left(1+\ln \overline{f}_{rw}^{\text{sue}}\right)+c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right),\cdots\right]^{\mathrm{T}}$$

where $c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)=\sum_{a\in A}t_a\left(v_a^{\text{sue}}\right)\delta_{ar}$, into Eq. (14), and in view of

$$\sum_{w\in W}\sum_{r\in R_w}\frac{1}{\theta}\left(f_{rw}-f_{rw}^{\text{sue}}\right)=\sum_{w\in W}\frac{1}{\theta}\left(\lambda_w d_w-\lambda_w d_w\right)=0$$

and

$$\sum_{w\in W}\sum_{r\in R_w}\frac{1}{\overline{\theta}}\left(\overline{f}_{rw}-\overline{f}_{rw}^{\text{sue}}\right)=\sum_{w\in W}\frac{1}{\overline{\theta}}\left((1-\lambda_w)d_w-(1-\lambda_w)d_w\right)=0$$

we then get the VI (12). □

According to the class composition of flows, we decompose the SO path flow $\mathbf{f}^{\text{so}}=\left(f_{rw}^{\text{so}},r\in R_w,w\in W\right)^T$ into $h_{rw}^{\text{so}}=\lambda_w f_{rw}^{\text{so}}$, $\overline{h}_{rw}^{\text{so}}=\left(1-\lambda_w\right)f_{rw}^{\text{so}}$, $r\in R_w$, $w\in W$. Clearly, $\mathbf{h}^{\text{so}}+\overline{\mathbf{h}}^{\text{so}}=\mathbf{f}^{\text{so}}$ and $\left(\mathbf{h}^{\text{so}},\overline{\mathbf{h}}^{\text{so}}\right)\in\Omega_{f,\overline{f}}$. From Lemma 1, we then have

$$\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)+\frac{1}{\theta}\ln f_{rw}^{\text{sue}}\right)\left(h_{rw}^{\text{so}}-f_{rw}^{\text{sue}}\right)$$
$$+\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)+\frac{1}{\overline{\theta}}\ln \overline{f}_{rw}^{\text{sue}}\right)\left(\overline{h}_{rw}^{\text{so}}-\overline{f}_{rw}^{\text{sue}}\right)\geq 0 \tag{15}$$

This leads to

$$T_{\text{mixed}}^{\text{sue}}\leq T^{\text{so}}+\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{\text{sue}}-c_{rw}^{\text{so}}\right)\left(h_{rw}^{\text{so}}+\overline{h}_{rw}^{\text{so}}\right)$$
$$+\sum_{w\in W}\sum_{r\in R_w}\frac{1}{\theta}\ln f_{rw}^{\text{suo}}\left(h_{rw}^{\text{so}}-f_{rw}^{\text{sue}}\right)+\sum_{w\in W}\sum_{r\in R_w}\frac{1}{\overline{\theta}}\ln \overline{f}_{rw}^{\text{sue}}\left(\overline{h}_{rw}^{\text{so}}-\overline{f}_{rw}^{\text{sue}}\right) \tag{16}$$

where

$$c_{rw}^{\text{sue}}=c_{rw}\left(\mathbf{f}^{\text{sue}},\overline{\mathbf{f}}^{\text{sue}}\right)$$

$$c_{rw}^{\text{so}}=c_{rw}\left(\mathbf{f}^{\text{so}}\right)=c_{rw}\left(\mathbf{h}^{\text{so}}+\overline{\mathbf{h}}^{\text{so}}\right)$$

$$T_{\text{mixed}}^{\text{sue}}=\sum_{w\in W}\sum_{r\in R_w}c_{rw}^{\text{sue}}\times\left(f_{rw}^{\text{sue}}+\overline{f}_{rw}^{\text{sue}}\right)=\sum_{a\in A}t_a\left(v_a^{\text{sue}}\right)v_a^{\text{sue}}$$

$$T^{\text{so}}=\sum_{w\in W}\sum_{r\in R_w}c_{rw}^{\text{so}}\times\left(h_{rw}^{\text{so}}+\overline{h}_{rw}^{\text{so}}\right)=\sum_{a\in A}t_a\left(v_a^{\text{so}}\right)v_a^{\text{so}}$$

Note that the second term of the right-hand-side of Eq. (16) can be rewritten as

$$\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{\text{sue}}-c_{rw}^{\text{so}}\right)\left(h_{rw}^{\text{so}}+\overline{h}_{rw}^{\text{so}}\right)=\sum_{a\in A}\left(t_a\left(v_a^{\text{sue}}\right)-t_a\left(v_a^{\text{so}}\right)\right)v_a^{\text{so}}$$

Hence, Eq. (16) becomes

$$T_{\text{mixed}}^{\text{sue}}\leq T^{\text{so}}+\sum_{a\in A}\left(t_a\left(v_a^{\text{sue}}\right)-t_a\left(v_a^{\text{so}}\right)\right)v_a^{\text{so}}$$
$$+\sum_{w\in W}\sum_{r\in R_w}\frac{1}{\theta}\ln f_{rw}^{\text{suo}}\left(h_{rw}^{\text{so}}-f_{rw}^{\text{sue}}\right)+\sum_{w\in W}\sum_{r\in R_w}\frac{1}{\overline{\theta}}\ln \overline{f}_{rw}^{\text{sue}}\left(\overline{h}_{rw}^{\text{so}}-\overline{f}_{rw}^{\text{sue}}\right) \tag{17}$$

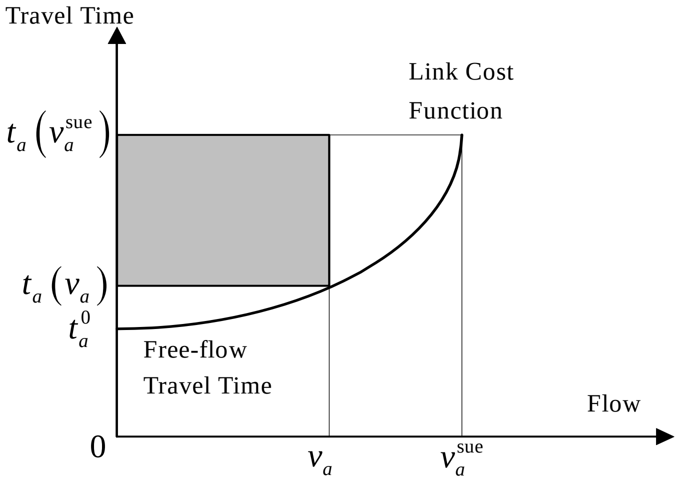

Fig. 1 Geometric illustration of the definition of $\gamma\left(C\right)$

We now consider how to determine the upper bound of the second term of the right-hand-side of Eq. (17). See the shaded rectangle in Fig. 1, which is $\left(t_a\left(v_a^{\text{sue}}\right)-t_a\left(v_a\right)\right)v_a$. For each link cost function $t_a=t_a\left(z_a\right)$ and nonnegative link flow $z_a\geq 0$, we define the following parameter (Yang and Huang, 2005)

$$\gamma_a\left(t_a,z_a\right)=\max_{v_a\geq 0}\frac{\left(t_a\left(z_a\right)-t_a\left(v_a\right)\right)v_a}{t_a\left(z_a\right)z_a} \tag{18}$$

Here, let $0/0=0$ by convention. Since $\left(t_a\left(z_a\right)-t_a\left(v_a\right)\right)v_a\leq t_a\left(z_a\right)v_a$ if $0\leq v_a\leq z_a$, and $\left(t_a\left(z_a\right)-t_a\left(v_a\right)\right)v_a<0$ if $v_a>z_a$, we then have $\gamma_a\left(t_a,z_a\right)\leq 1$. So, there exists an upper bound for $\gamma_a\left(t_a,z_a\right)$ with given link cost function t_a and link flow z_a. For a given class C of link cost function (for example, a family of linear cost function or polynomials of a certain degree), we let

$$\gamma\left(C\right)=\max_{t_a\in C,z_a\geq 0}\gamma_a\left(t_a,z_a\right) \tag{19}$$

With this definition, letting v_a be v_a^{so} and z_a be v_a^{sue} in Eq. (18), we then have

$$\sum_{a\in A}\left(t_a\left(v_a^{\text{sue}}\right)-t_a\left(v_a^{\text{so}}\right)\right)v_a^{\text{so}}\leq\gamma\left(C\right)\sum_{a\in A}t_a\left(v_a^{\text{sue}}\right)v_a^{\text{sue}}=\gamma\left(C\right)T_{\text{mixed}}^{\text{sue}} \tag{20}$$

Now we turn to seek a bound for the sum of the third and forth terms of the right-hand-side of Eq. (16) or Eq. (17), and thereby bound the overall cost inefficiency of the two-user class SUE. Before doing this, we first introduce a lemma which has been proven in Guo and Yang (2005) and Yang and Huang (2005).

Lemma 2 *Consider the following maximization problem*

$$\max\ F(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} (y_i - x_i) \ln x_i \tag{21}$$

subject to

$$\sum_{i=1}^{n} x_i = d\,,\ \ \sum_{i=1}^{n} y_i = d\,,\ \ x_i, y_i \ge 0\,,\ \ i = 1, 2, \cdots, n \tag{22}$$

where $d > 0$ *is a constant. The optimal objective function value of this problem is* $F_{\max} = kd$ *, where* k *solves such an equation that* $ke^k = (n-1)/e$ *, with* e *being the base of natural logarithm, or if it is defined that* $g(x) = xe^x$ *, then* $k = g^{-1}((n-1)/e)$*.*

From Lemma 2, it follows immediately that

$$\sum_{r\in R_w} \ln f_{rw}^{\text{suo}} \left(h_{rw}^{\text{so}} - f_{rw}^{\text{sue}} \right) \le k_w \lambda_w d_w \tag{23}$$

$$\sum_{w\in W} \sum_{r\in R_w} \ln \bar{f}_{rw}^{\text{sue}} \left(\bar{h}_{rw}^{\text{so}} - \bar{f}_{rw}^{\text{sue}} \right) \le k_w (1-\lambda_w) d_w \tag{24}$$

where $k_w = g^{-1}((|R_w|-1)/e)$, $w \in W$, and $|R_w|$ is the number of feasible paths between OD pair $w \in W$.

Substituting Eqs. (20), (23) and (24) into Eq. (17) yields

$$T_{\text{mixed}}^{\text{sue}} \le T^{\text{so}} + \gamma(C) T_{\text{mixed}}^{\text{sue}} + \frac{1}{\theta} \sum_{w\in W} k_w \lambda_w d_w + \frac{1}{\bar{\theta}} \sum_{w\in W} k_w (1-\lambda_w) d_w \tag{25}$$

Let $D = \sum_{w\in W} d_w$ be the total traffic demand in the network; $\kappa = \sum_{w\in W} (\lambda_w d_w / D) k_w$ be the equipped drivers' average of k_w , weighted by total OD demand; and $\bar{\kappa} = \sum_{w\in W} ((1-\lambda_w) d_w / D) k_w$ be the unequipped drivers' average of k_w, weighted by total OD demand. Then, Eq. (25) can be rewritten as

$$T_{\text{mixed}}^{\text{sue}} \le T^{\text{so}} + \gamma(C) T_{\text{mixed}}^{\text{sue}} + \left(\frac{\kappa}{\theta} + \frac{\bar{\kappa}}{\bar{\theta}} \right) D \tag{26}$$

Furthermore, define $\tilde{c} = T^{\text{so}}/D$ as the average travel time of all network users at system optimum. Then, from Eq. (26), we arrive at the following bounding result.

Theorem 1 *Let* C *be a family of continuous, nondecreasing latency functions. Consider an instance of the two-user class SUE model (3)-(8) with separable link cost functions drawn from* C*. Then, the ratio of the total travel time of the two-user class SUE flow* *pattern to that of a system optimum is bounded from above by* $\rho_{\text{mixed}}^{\text{sue}}$, *i.e.,*

$$\rho_{\text{mixed}}^{\text{sue}} = \frac{T_{\text{mixed}}^{\text{sue}}}{T^{\text{so}}} \le \left(\frac{1}{1-\gamma\left(C\right)} \right) \left(1 + \frac{1}{\tilde{c}} \left(\frac{\kappa}{\theta} + \frac{\overline{\kappa}}{\overline{\overline{\theta}}} \right) \right) \tag{27}$$

We note that the inefficiency bound given in Theorem 1 is a worst-case measure, taken over all possible instances. The actual ratio $\rho_{\text{mixed}}^{\text{sue}}$ in realistic instances could be substantially smaller. Indeed, in a traffic network, the free-flow travel time is usually not a negligible fraction, which is encountered in both SUE and SO flow states. Here, similar to Correa et al. (2005) and Yang and Huang (2005), we present a parameterized, improved bound on the inefficiency of SUE.

Theorem 2 *Let* C *be a family of continuous, nondecreasing latency functions. Consider an instance of the two-user class SUE model (3)-(8) with separable link cost functions drawn from* C, *such that the free-flow link travel time* $t_a\left(0\right) \ge \eta\left(\mathbf{v}^{\text{sue}}\right) t_a\left(v_a^{\text{sue}}\right)$ *for all* $a \in A$, *for some positive constant* $0 \le \eta\left(\mathbf{v}^{\text{sue}}\right) \le 1$, *where* η *can depend on the SUE link flow pattern* $\mathbf{v}^{\text{sue}}$. *Then, the ratio of the total travel time of the two-user class SUE flow to that of a system optimum is bounded from above by* $\rho_{\text{mixed}}^{\text{sue}}$, *i.e.,*

$$\rho_{\text{mixed}}^{\text{sue}} = \frac{T_{\text{mixed}}^{\text{sue}}}{T^{\text{so}}} \le \left(\frac{1}{1-\left(1-\eta\left(\mathbf{v}^{\text{sue}}\right)\right)\gamma\left(C\right)} \right) \left(1 + \frac{1}{\tilde{c}} \left(\frac{\kappa}{\theta} + \frac{\overline{\kappa}}{\overline{\overline{\theta}}} \right) \right) \tag{28}$$

4 Parameters influencing the bound

As shown in Eq. (28), the cost inefficiency bound of the two-user class SUE depends on seven parameters, namely $\gamma\left(C\right)$, $\eta\left(\mathbf{v}^{\text{sue}}\right)$, $\tilde{c}$, κ, $\overline{\kappa}$, θ and $\overline{\theta}$. We now investigate the properties of these parameters and examine their effects on the cost inefficiency bound.

$\gamma\left(C\right) \le 1$ is a dimensionless number defined exclusively by the class of link cost functions. The bound increases with the value of $\gamma\left(C\right)$. Consider the special case of polynomial link cost functions, $t_a(v_a)=t_a^0+\alpha_a(v_a)^p$, $a\in A$, where t_a^0 is a constant representing free-flow link travel time, $\alpha_a\ge 0$ is a link-specific parameter and $p\ge 0$ is a real number. Yang and Huang (2005) showed that, with the polynomial link cost functions, $\gamma(C)$ has an upper bound

$$\gamma(C)\le\left(\frac{p}{p+1}\right)\left(\frac{1}{p+1}\right)^{1/p} \tag{29}$$

When p=1 and 4, $\gamma(C)$=0.25 and 0.535 respectively. In fact, as $p\to 0$, i.e., without traffic congestion in the network, $\gamma(C)\to 0$; and as $p\to+\infty$, i.e., with serve congestion in the network, $\gamma(C)\to 1$. Hence, the bound is positively proportional to the degree of traffic congestion in the network. This says, in a highly congested (not caused by the traffic demand but the parameters of link cost functions) network, there exists more space for system improvement when driving the SUE state to a SO state.

Eq. (28) shows that, when $\eta\left(\mathbf{v}^{\text{sue}}\right)$ approaches 1, the bound decreases. Hence, the more the free-flow times of all links approach the link travel times generated in SUE state, the more the bound declines (i.e., existing less space for system improvement when driving the SUE to a SO).

Eq. (28) also shows that the bound decreases with the average travel time of all network users at SO. Let the average free-flow times for all users be $\tilde{c}_0$, clearly $\tilde{c}_0\le\tilde{c}$. Then, we can relax Eq. (28) by replacing $\tilde{c}$ with $\tilde{c}_0$. In this sense, the bound increases with the total traffic demand.

Recall that $k_w=g^{-1}\left(\left(|R_w|-1\right)/e\right)$, $\kappa=\sum_{w\in W}\left(\lambda_w d_w/D\right)k_w$, $\bar{\kappa}=\sum_{w\in W}\left((1-\lambda_w)d_w/D\right)k_w$. Hence, both κ and $\bar{\kappa}$ are dimensionless coefficients that increase with the number of feasible paths, and thus reflect the degree of network complexity. If there is only one path available for each OD pair in the network, i.e., $|R_w|=1, w\in W$, then $k_w=0$, $w\in W$ and thus $\kappa=\bar{\kappa}=0$. In this case, we have $\rho_{\text{mixed}}^{\text{sue}}\le\left(1-\left(1-\eta\left(\mathbf{v}^{\text{sue}}\right)\right)\gamma(C)\right)^{-1}$. As demonstrated in Guo and Huang (2005), whenever $|R_w|>1$, the value of k_w (and hence κ and $\bar{\kappa}$) is very limited. For sufficiently complex network with the number of paths between each OD pair, $100 \le |R_w| \le 1000$, k_w takes only a limited value between 2.63 and 4.2. This observation shows that the network topology, or the network size and complexity, has very limited effects on the cost efficiency bound of the SUE, as claimed by Yang and Huang (2005). Even though the effect is small, Eq. (28) shows that the bound would increase when the degree of network complexity goes up.

Obviously, if $\lambda_w = 0$ or 1, the cost inefficiency bound derived in this study is the same with that given by Yang and Huang (2005) in which only one user class is considered. If all drivers are not equipped with ATIS, i.e., $\lambda_w = 0$, $w \in W$, then $\kappa = 0$ and thus $\rho_{\text{mixed}}^{\text{sue}}(\text{all unequipped}) \le \left(\frac{1}{1-\gamma(C)}\right)\left(1+\frac{1}{\tilde{c}}\times\frac{\bar{\kappa}}{\bar{\theta}}\right)$ where $\bar{\kappa} = \sum_{w\in W}(d_w/D)k_w$; if all drivers are equipped with ATIS, i.e., $\lambda_w = 1$, $w \in W$, then $\bar{\kappa} = 0$ and thus $\rho_{\text{mixed}}^{\text{sue}}(\text{all equipped}) \le \left(\frac{1}{1-\gamma(C)}\right)\left(1+\frac{1}{\tilde{c}}\times\frac{\kappa}{\theta}\right)$ where $\kappa = \sum_{w\in W}(d_w/D)k_w$. As $\theta > \bar{\theta}$, thus the bound reached in a ATIS fully adopted network is less than that in a without-ATIS network. Furthermore, we rewrite Eq. (27) as

$$\rho_{\text{mixed}}^{\text{sue}} = \frac{T_{\text{mixed}}^{\text{sue}}}{T^{\text{so}}} \le \left(\frac{1}{1-\gamma(C)}\right)\left(1+\frac{1}{\tilde{c}}\left(\frac{1}{\bar{\theta}}\sum_{w\in W}\left(\frac{d_w}{D}\right)k_w + \left(\frac{1}{\theta}-\frac{1}{\bar{\theta}}\right)\sum_{w\in W}\left(\frac{\lambda_w d_w}{D}\right)k_w\right)\right) \quad (30)$$

This shows that, with given θ and $\bar{\theta}$, the bound decreases with the ATIS market penetration λ_w since $\theta > \bar{\theta}$.

We now check the effect of ATIS quality on the bound. We know that the logit model parameter θ is inversely proportional to the standard error of the perceived travel time of equipped drivers, so, the larger the θ-value, the higher the information quality provided by ATIS. Let the $\bar{\theta}$-value of unequipped drivers be fixed, Eq. (27) shows that the bound decreases with the θ-value. This says, the bound would decrease when the ATIS quality is improved. As $\theta \to \infty$, we have $\rho_{\text{mixed}}^{\text{sue}} = \frac{T_{\text{mixed}}^{\text{sue}}}{T^{\text{so}}} \le \left(\frac{1}{1-\gamma(C)}\right)\left(1+\frac{1}{\tilde{c}}\times\frac{\bar{\kappa}}{\bar{\theta}}\right)$ where $\bar{\kappa} = \sum_{w\in W}\left((1-\lambda_w)d_w/D\right)k_w$. We hereby get the price of anarchy of the mixed UE and SUE problem with the proportion of number of UE users over all, λ_w between OD pair

$w \in W$. Furthermore, as $\bar{\theta} \to \infty$ too, we have $\rho_{\text{mixed}}^{\text{sue}} = \dfrac{1}{1-\gamma(C)}$ which is the result of the price of anarchy of the standard UE problem, independent of the market penetration level.

If all drivers are homogeneous, i.e., the ATIS doesn't differentiate drivers at all, $\theta = \bar{\theta}$, we then have $\rho_{\text{mixed}}^{\text{sue}} \le \left(\dfrac{1}{1-\gamma(C)}\right)\left(1+\dfrac{1}{\tilde{c}} \times \dfrac{\bar{\kappa}}{\bar{\theta}}\right)$ where $\bar{\kappa} = \sum_{w \in W} \left(d_w / D\right) k_w$. This is the result of the standard SUE problem, identical to Yang and Huang (2005), independent of the market penetration level.

Finally, we use a table to summarize the effects of all parameters on the cost inefficiency bound, see Table 1.

Table 1 Effects of various parameters on the cost inefficiency bound

	Congestion degree	Total demand	Network complexity	ATIS market penetration	Information quality
Bound	+	+	+	–	–

5 Conclusions

We have derived the cost inefficiency bound of a stochastic user equilibrium where all users are divided into two classes, one equipped with ATIS and another unequipped. Users of both classes follow the SUE principle in route choices, the equipped being able to receive real-time traffic information for helping route choice and thus having less degree of travel time variability than unequipped. We have further investigated the effects of various parameters which reflect the network's complexity, the link cost function's property, the ATIS quality and market penetration, on the bound of the cost inefficiency of the two-user class SUE flow pattern in relation to the Wardropian system optimum. The results are summarized in Table 1.

Some issues that we are further investigating are: *(a)* to extend the analysis to the case with elastic demand; *(b)* to study the two-user class SUE model with link capacity constraints; and *(c)* to seek more accurate bound for some specific classes of link cost functions.

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